

On conformal higher spins in curved background

M. Grigoriev^a and A.A. Tseytlin^{a,b}

^a*Tamm Theory Department, Lebedev Physics Institute
Leninsky prospect 53, 119991 Moscow, Russia*

^b*Theoretical Physics Group, Blackett Laboratory,
Imperial College London, SW7 2AZ, U.K.*

Abstract

We address the question of how to represent the interacting action for a tower of conformal higher spin fields in a form covariant with respect to a background metric. We use the background metric to define a star product which plays a central role in the definition of the corresponding gauge transformations. By analogy with the kinetic term in the 4-derivative Weyl gravity action expanded near an on-shell background, one expects that the kinetic term in such an action should be gauge-invariant in a Bach-flat metric. We demonstrate this fact to first order in expansion in powers of the curvature. This generalizes the result of arXiv:1404.7452 for spin 3 case to all conformal higher spins. We also comment on a possibility of extending this claim to terms quadratic in the curvature and discuss the appearance of background-dependent mixing terms in the quadratic part of the conformal higher spin action.

Contents

1	Introduction	2
2	Particle Hamiltonian in CHS background and expansion near flat space	5
2.1	Gauge transformations	6
2.2	Conserved currents	7
3	Covariant form of the scalar field action in CHS background	8
4	Covariant expansion of the CHS action in a non-trivial metric	10
4.1	Expansion of the CHS action	10
4.2	Conditions for the vanishing of the linear fluctuation term	12
4.3	Gauge invariance of spin- s quadratic term to first order in curvature	13
4.4	Spin 3 example	14
5	Conclusions	16
A	Covariant quantization in Fedosov-type approach: quantum version of normal coordinate expansion	17
B	Weyl invariants	20

1 Introduction

Conformal higher spin (CHS) field models are $s > 2$ generalizations of Maxwell ($F_{\mu\nu}^2$) and Weyl ($C_{\mu\nu\lambda\rho}^2$) theories [1, 2, 3, 4]. While they have higher-derivative ∂^{2s} kinetic terms and thus are formally non-unitary they have a remarkable feature of describing pure spin s states off-shell, i.e. have maximal spin s gauge symmetry consistent with locality.

The free CHS action in flat 4-dimensional space may be written as

$$S_s = \int d^4x h_s P_s \partial^{2s} h_s = \int d^4x (-1)^s C_s C_s, \quad (1.1)$$

where $h_s = (h_{\mu_1 \dots \mu_s})$ is a totally symmetric tensor and $P_s = (P_{\nu_1 \dots \nu_s}^{\mu_1 \dots \mu_s})$ is the transverse projector which is traceless and symmetric within μ and ν groups of indices. This action is thus invariant under a combination of differential (generalized reparametrizations) and algebraic (generalized

Weyl) gauge transformations: $\delta h_s = \partial \epsilon_{s-1} + \eta_2 \omega_{s-2}$ (here η_2 is a flat metric and ϵ and ω are parameter tensors). $C_s = (C_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s})$ is the gauge-invariant field strength or generalized Weyl tensor.

The theory containing an infinite tower of CHS fields h_s ($s = 0, 1, 2, \dots$) is a non-trivial interacting field theory with an action that can be defined as a local part of an induced action [3, 4, 5, 6]. Explicitly, one may start with a free CFT of N scalar fields $\int d^4x \phi_i^* \partial^2 \phi_i$ which has the on-shell conserved and traceless spin s currents $J_s = \phi_i^* \mathcal{J}_s \phi_i$ ($\mathcal{J}_s \sim \partial^s + \dots$) and consider the generating functional for correlation functions of these currents

$$\Gamma[h] = N \log \det \Delta(h), \quad \Delta(h) = -\partial^2 + \sum_s h_s \mathcal{J}_s. \quad (1.2)$$

Here $h_s(x)$ are source fields which have linearized gauge symmetries implied by the on-shell conservation and tracelessness of the currents J_s .¹ The UV logarithmically divergent part of (1.2) is local, has the required linearized gauge symmetries and expanded in h_s starts with (1.1) as its quadratic term.² The coefficient of the logarithmic divergence (or, equivalently, the t^0 Seeley coefficient in the small t expansion of the heat kernel of the operator $\Delta(h)$) can thus be taken as a definition of the full CHS action, i.e.

$$S[h] = N [\log \det \Delta(h)]_{\log \Lambda} \sim N \text{Tr} e^{-t\Delta(h)} \Big|_{t=0}. \quad (1.3)$$

In this particular construction N plays the role of the square of the inverse coupling constant which, in general, can be arbitrary. A discussion of some cubic and quartic terms in this action appeared in [5, 7, 8].³

This CHS theory has a close connection to AdS/CFT but has also several remarkable features on its own. On general grounds, the theory $S_{\text{CHS}} \sim \int d^4x (h_0^2 + F_{\mu\nu}^2 + C_{\mu\nu\lambda\rho}^2 + \dots)$ with dimensionless coupling constant should be renormalizable – the gauge symmetries should fix the local action uniquely. The central question is the absence of anomalies, in particular, the Weyl anomaly. It was found in [6, 9] that the one-loop a -coefficient of Weyl anomaly of the $d = 4$ CHS theory vanishes under a particular prescription (which should be consistent with the underlying symmetries, see also [10, 11]) for summation over spins. The same was found also for the one-loop conformal anomaly c -coefficient [9, 10, 12, 11] under the assumption that contributions to

¹According to vectorial AdS/CFT this induced action should follow also from the massless higher spin theory in the AdS_5 bulk upon computing it on the solution of the equations of motion with h_s setting the boundary conditions for the 5d massless higher spin fields.

²One gets $\Gamma[h] = N \sum_s \int h_s K_s h_s + O(h^3)$, where $K_s \sim N^{-1} \langle J_s(x) J_s(x') \rangle \sim P_s |x - x'|^{-4-2s} \sim P_s \partial^{2s} \delta^{(4)}(x - x') \log \Lambda + \dots$. Let us note that to get diagonal kinetic terms for all CHS fields one needs to apply a certain field redefinition required to make the algebraic Weyl-type symmetry manifest [4].

³Since the dimension of h_s is $2 - s$ and the theory is scale invariant the h^m ($m = 3, 4, \dots$) interaction vertex containing fields of spins s_i ($i = 1, \dots, m$) involves $k = 4 + \sum_{i=1}^m (s_i - 2)$ derivatives. Thus the coupling to the dimension 0 spin 2 field (conformal graviton) is special: one may add an arbitrary number of h_2 factors in the vertex without increasing the number of derivatives.

the conformal anomaly from higher derivative CHS operators on Ricci flat background factorize.⁴ As the Weyl symmetry is one of the CHS gauge symmetries, this is an indication that the same anomaly cancellation may apply to all algebraic CHS symmetries.

The CHS theory has also the vanishing Casimir energy on $R \times S^3$ [14] and zero total number of degrees of freedom (trivial flat-space partition function) [11] which is a reflection of the large underlying gauge symmetry of this theory. The global part of this symmetry also strongly constrains the S-matrix involving exchanges of the CHS fields implying that it should be trivial [7, 8].

The action (1.3) is naturally defined as an expansion in powers of h_s fields near flat space. It can thus be interpreted as a higher spin interacting classical conformal field theory. One may then wonder if it may admit a reparametrization and Weyl covariant generalization to a curved background which is known to exist for the standard low-spin ($s = 1, 2$) cases. Assuming that the $s = 2$ field $h_{\mu\nu}$ may be interpreted as the conformal graviton, one may ask if the action (1.3) can be rewritten (after some field redefinitions) as an expansion near a curved background $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$.

Here we will restrict attention to terms in the CHS action that are quadratic in the fields h_s but all-order in the background metric $g_{\mu\nu}$ and address the question which background geometries admit a consistent (gauge-covariant) propagation of h_s . It follows from the flat-space conformal invariance that the CHS field can be consistently defined on any conformally flat background $g_{\mu\nu}(x) = \sigma(x)\eta_{\mu\nu}$. In the case of an arbitrary $\sigma(x)$ the form of the generic spin s kinetic operator is not known explicitly but can be reconstructed, in principle, by a σ -dependent rescaling of the field (assuming there exists a Weyl-invariant generalization of the flat-space action (1.1)).⁵ In the case of a homogeneous conformally-flat space (4-sphere or AdS or dS or $R \times S^3$) the CHS kinetic operator is known and can be represented as a product of second-order differential operators [9, 15, 16, 13, 14]. The question is whether the CHS fields can be consistent on non-conformally-flat backgrounds with non-vanishing Weyl tensor and what are the conditions on the Weyl tensor for this to happen.

For $s = 1$ (Maxwell) and $s = 2$ (Weyl) cases the CHS kinetic terms admit the well-known generalizations to a non-trivial background metric $g_{\mu\nu}$. For $s = 1$ we get no constraints on $g_{\mu\nu}$ while for $s = 2$ the invariance of the quadratic term in the Weyl action $\int d^4x \sqrt{g} C_{\mu\nu\lambda\rho}^2$ expanded

⁴The computation of the one-loop conformal anomaly c -coefficient in the CHS theory in [9] was based on two assumptions: (i) the CHS action obtained as a UV divergent part of the induced action in near-flat space expansion can be reformulated (using a field redefinition) in such a way that at least quadratic kinetic terms in generic curved metric background are reparametrization and Weyl covariant; (ii) the higher derivative kinetic operators $\nabla^{2s} + \dots$, while not factorizing, in general, into products of $\nabla^2 + \dots$ operators on a Ricci-flat background [13] (as they do on *AdS* or on the sphere) still contribute to the c -anomaly in the same way as if they were factorizing. The reason is that the terms with derivatives of the curvature tensor that obstruct the factorization can not contribute to the $C_{\mu\nu\kappa\iota}^2$ conformal anomaly on dimensional grounds.

⁵Alternatively, including some auxiliary and Stueckelberg fields one can reformulate the CHS action in a manifestly conformal form for which rewriting in a generic conformally-flat background amounts to just picking an appropriate $o(d, 2)$ -connection and conformal compensator.

in $h_{\mu\nu}$ (with $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$) gives a special $\nabla^4 + \dots$ kinetic operator [17, 1, 14]. This operator is covariant under the gauge transformations $\delta h_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu + \omega g_{\mu\nu}$ provided $g_{\mu\nu}$ is an on-shell background for the Weyl theory, i.e. is Bach-flat,

$$B_{\mu\nu} = 0, \quad B_{\mu\nu} \equiv \nabla^\rho \nabla_\mu P_{\nu\rho} - \nabla^\rho \nabla_\rho P_{\mu\nu} + P^{\rho\sigma} C_{\mu\rho\nu\sigma}, \quad P_{\mu\nu} \equiv \frac{1}{2} (R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu}). \quad (1.4)$$

For $s = 3$ CHS field this question was addressed in [13] where the corresponding covariant $\nabla^6 + \dots$ kinetic operator was found to linear order in the background curvature tensor and was shown to be gauge-invariant on Bach-flat backgrounds (to first order in the curvature).

A goal of the present paper is to make a step towards a covariant description of all CHS fields on curved Bach-flat (or, in particular, Ricci-flat) backgrounds. Our starting point will be an equivalent definition of the non-linear CHS action (1.3) based on an effective particle Hamiltonian associated with the operator $\Delta(h)$ in (1.2) [4] that makes the full non-linear symmetry of the theory more explicit.

In section 2 we shall review the definition of the particle Hamiltonian coupled to the CHS fields following [4, 5]. Its quantization leads to a quadratic scalar action in CHS background that has gauge invariances inherited from the freedom in the definition of the particle dynamics. In section 3 we shall suggest a procedure of how to define the scalar action in a way covariant with respect to a background metric and having the required gauge symmetries. Then the corresponding CHS action can be again defined as a UV singular part of the induced action found after integrating out the scalar field.

In section 4 we shall analyse the expansion of this action in powers of the CHS fields and the consistency conditions of this expansion using perturbation theory in powers of the curvature of the background metric. Section 5 will contain some concluding remarks. In Appendix A we shall review the Fedosov-type approach to covariant formulation of first-quantized particle dynamics that plays important role in our definition of the CHS gauge transformations in a non-trivial background. In Appendix B we shall make some general comments on the structure of Weyl invariants built out of the curvature and its covariant derivatives.

2 Particle Hamiltonian in CHS background and expansion near flat space

Before developing a covariant approach to CHS fields let us briefly recall how their gauge transformations and gauge-invariant action arise from the coordinate-dependent quantized particle approach [4, 18].

2.1 Gauge transformations

Let us start with a space-time manifold with coordinates x^μ and introduce the momenta p_μ conjugate to x^μ . We will interpret functions of (x, p) which are smooth in x and polynomial in p as symbols of differential operators acting on “wave functions” of x . The $*$ -product will denote the operator composition in terms of (Weyl) symbols⁶

$$* = \exp \left[\frac{\hbar}{2} \left(\overleftarrow{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu} \frac{\partial}{\partial x^\mu} \right) \right]. \quad (2.1)$$

Let us consider a generic relativistic particle Hamiltonian generalizing the free one H_0

$$H(x, p) = H_0(x, p) + h(x, p), \quad h(x, p) = \sum_{s=0}^{\infty} h^{\mu_1 \dots \mu_s}(x) p_{\mu_1} \dots p_{\mu_s} \quad (2.2)$$

and subject it to the following gauge transformations [18, 4]⁷

$$\delta H = \hbar^{-1} [H, \epsilon(x, p)]_* + \{H, \omega(x, p)\}_* , \quad (2.3)$$

where ϵ, ω are unconstrained symbols interpreted as gauge parameters.⁸ They induce the linearized CHS gauge transformations of the coefficient fields h_s .

These gauge symmetries have a simple interpretation [19, 20] in the context of a constrained system $T_a(x, p) = 0$ where the symbols $T_a(x, p)$ are subject to the 1-st class condition $[T_a, T_b]_* = U_{ab}^c * T_c$. A given constrained system can be described by an equivalent set of constraints: an infinitesimal equivalence relation $T_a \sim T_a + \hbar^{-1} [T_a, \chi]_*$ corresponds to an infinitesimal canonical transformation while the equivalence relation $T_a \sim T_a + \lambda_a^b * T_b$ corresponds to an infinitesimal redefinition of the constraints (which preserves the constraint surface). Then the space of gauge-inequivalent configurations is a moduli space of constrained systems that have fixed number of 1-st class constraints and satisfy certain extra conditions (e.g. belong to a vicinity of certain vacuum H_0).⁹

To relate this to (2.3) let us consider the case of just one constraint $T \equiv H$ and identify parameters as $\epsilon = \chi - \frac{\hbar}{2} \lambda$, $\omega = \frac{1}{2} \lambda$. Then the gauge transformations (2.3) are the natural equivalence transformations of the constrained system describing a relativistic particle. The “vacuum” (quadratic in p) choice of H

$$H_0 = g(x, p) \equiv -\frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu \quad (2.4)$$

⁶Here \hbar is a formal parameter that can be always set to 1.

⁷Here the commutator $[\ , \]_*$ and anticommutator $\{ \ , \ \}_*$ are defined with respect to the above $*$ -product.

⁸This gauge symmetry is reducible: $\delta\epsilon = \{H, \alpha\}_*$, $\delta\omega = -[H, \alpha]_*$ where α is a reducibility parameter.

⁹In the BRST description the constraints are encoded in the symbol $\Omega(x, p, \text{ghosts})$ while the above equations and gauge transformations are encoded in $[\Omega, \Omega]_* = 0$, $\delta\Omega = \hbar^{-1} [\Omega, \Xi]_*$ where Ξ contains χ and λ_b^a and $[\ , \]_*$ is a graded commutator. In the context of string field theory such sort of systems were considered in [21].

is the standard Hamiltonian of a particle in a gravitational background. In this case the linearized gauge transformations (2.3) read as

$$\begin{aligned}\delta h &= \hbar^{-1} [H_0, \epsilon(x, p)]_* + \{H_0, \omega(x, p)\}_* \\ &= p_\mu g^{\mu\nu} \frac{\partial}{\partial x^\nu} \epsilon - p_\mu \left(\frac{1}{2} \partial_\rho g^{\mu\nu} \right) p_\nu \frac{\partial}{\partial p_\rho} \epsilon - \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \omega + O(\hbar). \quad (2.5)\end{aligned}$$

Let $\hat{H}(x, \frac{\partial}{\partial x})$ be a differential operator associated to the symbol $H(x, p)$ (assumed to be such that \hat{H} is formally hermitian). As we are using the Weyl symbols this means that H is real if x^μ is real and p_μ is imaginary. Then the complex scalar action defined as¹⁰

$$S[\phi, h] = \int d^d x \phi^*(x) \hat{H}(x, \frac{\partial}{\partial x}) \phi(x) \quad (2.6)$$

is invariant under the transformations (2.3) provided at the same time ϕ transforms as¹¹

$$\delta \phi = -(\hbar^{-1} \hat{\epsilon} + \hat{\omega}) \phi, \quad \epsilon^\dagger = -\epsilon, \quad \omega^\dagger = \omega. \quad (2.7)$$

As follows from (2.3) and properties of the Weyl star product, one can consistently put to zero all fields h_s of odd spins appearing in (2.2) along with the gauge parameters ϵ / ω of even / odd degree in p_μ . On top of this there is a consistent truncation to a system where all fields with $s > 2$ and their associated gauge parameters are set to zero. This is due to the fact that the elements which are at most linear in p_μ form a Lie subalgebra of a Weyl star-product algebra. The above two truncations can be combined, resulting in a system for fields of spins 0 and 2 only. Furthermore, the spin 0 field can also be eliminated. Apart from the above consistent truncation to spins ≤ 2 one can not get a gauge invariant action depending only on a finite number of fields h_s .

2.2 Conserved currents

For H in (2.2), (2.4) the action (2.6) may be written as

$$S = \int d^d x \left[\phi^* \hat{H}_0 \phi + \phi^* \hat{h} \phi \right]. \quad (2.8)$$

The condition of its invariance under (2.3) combined with $\delta \phi = -\hbar^{-1} \hat{\epsilon} \phi$ takes the form

$$\int d^d x \left(([H_0, \epsilon]_* + [h, \epsilon]) \frac{\delta S}{\delta h} - \hat{\epsilon} \phi^* \frac{\delta S}{\delta \phi} \right) = 0. \quad (2.9)$$

Introducing a generating function for conserved currents (here u^μ is an auxiliary constant vector)

$$J = \sum_s \frac{1}{s!} u^{\mu_1} \dots u^{\mu_s} \frac{\delta S}{\delta h^{\mu_1 \dots \mu_s}}, \quad (2.10)$$

¹⁰ Here $*$ is complex conjugation which should not be confused with star product defined above.

¹¹ Note that by removing (anti)hermiticity conditions on $\hat{H}, \hat{\omega}, \hat{\epsilon}$ one finds the “equations of motion” version of the system. Indeed, in this case that the above gauge symmetries are symmetries of the following equations of motion: $\hat{H} \phi = 0$, $J \equiv \frac{\delta S[\phi]}{\delta H} = 0$.

the quadratic in the fields term in (2.9) may be written as

$$\int d^d x \left(\langle [H_0, \epsilon]_*, J \rangle - 2(\widehat{\epsilon}\phi)^* \widehat{H}_0 \phi \right) = 0, \quad (2.11)$$

where $\langle \cdot, \cdot \rangle$ denotes a natural inner product (contraction of indices) between polynomials in p_μ and polynomials in u^μ .¹² For $H_0 = -\frac{1}{2}\eta^{\mu\nu} p_\mu p_\nu \equiv -\frac{1}{2}p^2$ one gets the usual on-shell conservation condition for the currents

$$\eta^{\mu\nu} \frac{\partial}{\partial u^\mu} \frac{\partial}{\partial x^\nu} J = 0. \quad (2.12)$$

Applying analogous arguments to the second gauge invariance with the parameter ω in (2.3) results in the generalized on-shell tracelessness condition for the currents:

$$\int d^d x \left(\langle \{H_0, \omega\}, J \rangle - 2(\widehat{\omega}\phi)^* \widehat{H}_0 \phi \right) = 0. \quad (2.13)$$

For $H_0 = -\frac{1}{2}p^2$ one gets the “deformed” tracelessness condition

$$(\eta^{\mu\nu} \frac{\partial}{\partial u^\mu} \frac{\partial}{\partial u^\nu} - \frac{1}{2}\hbar^2 \square) J = 0. \quad (2.14)$$

Redefining the components of J one can make them strictly traceless [4] but it is not always useful to perform this redefinition explicitly.

More generally, taking a variational derivative of (2.9) with respect to ϵ and not decomposing the result according to the homogeneity in the fields leads to a nonlinear generalization of the conservation condition (2.12) which now involves the fields h_s . Analogous arguments apply to gauge invariance under the transformations with the parameter ω leading to a nonlinear version of (2.14).

3 Covariant form of the scalar field action in CHS background

To generalize the above discussion to a curved background we shall first consider a covariant framework for a relativistic particle quantization. One can naturally define a quantization of the cotangent bundle over a curved spacetime in a geometrically covariant way. This covariant description is based on the metric $g_{\mu\nu}$ and the metric connection and includes (see also Appendix A):

- *star product*: for any functions of x^μ, p_ν which are smooth in x and polynomial in p there is a well-defined (and unique under some extra natural conditions) associative $*$ -product. The $*$ -commutator (anti-commutator) carries odd (resp. even) homogeneity in p .¹³
- *state space*: space of functions of x^μ equipped with the natural inner product

$$\langle \phi, \chi \rangle = \int d^d x \sqrt{g} \phi^*(x) \chi(x). \quad (3.1)$$

¹²For instance, $\langle 1, 1 \rangle = 1$, $\langle u^\mu, p_\nu \rangle = \delta_\nu^\mu$, etc. Note that then $(u^\mu)^\dagger = \frac{\partial}{\partial p_\mu}$, etc.

¹³For example, if $f(x, -p) = f(x, p)$ and $u(x, -p) = u(x, p)$ then $[f, u]_*(x, -p) = -[f, u]_*(x, p)$.

- *symbol map*: there is a well-defined map $f \rightarrow \widehat{f}$ from functions of x, p (symbols) to differential operators acting on functions of x^μ (“wave functions”) such that

$$\widehat{f_1 * f_2} = \widehat{f_1} \widehat{f_2}. \quad (3.2)$$

The operator associated to $f(x, p)$ is denoted by $\mathfrak{R}(g, f) = \widehat{f}(x, \frac{\partial}{\partial x})$. Here g indicates the dependence on the background metric, i.e. on the covariant derivative and the curvature built out of it (see Appendix A and more specifically (A.13) and propositions A.2 and A.3). The real symbols correspond to hermitian operators.

Let us redefine the spin 0 part of $h(x, p)$ in (2.2) by the scalar curvature of the metric and write H in (2.2),(2.4) as

$$H(x, p) = g(x, p) + \mathcal{R}(x) + h(x, p), \quad g(x, p) = -\frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu, \quad \mathcal{R} = \gamma R, \quad (3.3)$$

where R is the scalar curvature and γ is a numerical coefficient. The covariant version of the scalar field action (2.6),(2.8) then reads

$$S[g, h, \phi] = \int d^d x \sqrt{g} \phi^* \left(\widehat{g(x, p)} + \mathcal{R}(x) \right) \phi + \int d^d x \sqrt{g} \phi^* \widehat{h(x, p)} \phi, \quad (3.4)$$

where we have explicitly separated the h -independent term. The coefficient γ in (3.3) is chosen so that the first term in the above action is the standard action of the conformally coupled scalar. Note that this action depends on $g_{\mu\nu}$ also through the metric connection entering the symbol map.

By construction, this action is invariant under the covariant version of (2.3) and (2.7)¹⁴

$$\delta h = [g + \mathcal{R} + h, \epsilon]_* + \{g + \mathcal{R} + h, \omega\}_*, \quad (3.5)$$

$$\delta \phi = -(\widehat{\epsilon + \omega}) \phi. \quad (3.6)$$

Let us stress that the background field $g_{\mu\nu}(x)$ is not affected by this gauge transformation. However, there are hidden gauge transformations of the action (3.4) related to redefinition of $g^{\mu\nu}$ and $h_2 = (h^{\mu\nu})$ which do not change their sum $g + h$ modulo relevant redefinition of the symbol map (which depends on $g^{\mu\nu}$). For further analysis it is useful to employ two extra types of symmetries which have natural geometrical meaning and are, in fact, certain combinations of (3.5),(3.6) and redefinitions of $g^{\mu\nu}$ and $h^{\mu\nu}$.

First, given that the action (3.4) contains only covariant objects, it is invariant under the diffeomorphisms generated by a vector field $\xi = \xi^\mu \frac{\partial}{\partial x^\mu}$ with $h^{\mu_1 \dots \mu_s}$ transforming as tensors, i.e. under

$$\delta g = \mathcal{L}_\xi g, \quad \delta h = \mathcal{L}_\xi h, \quad \delta \phi = \xi \phi. \quad (3.7)$$

Second, the h -independent term in (3.4) is the action of a conformally coupled scalar and hence it is invariant under the usual Weyl symmetry $\delta_{\omega_0} g^{\mu\nu} = 2\omega_0 g^{\mu\nu}$, $\delta_{\omega_0} \phi = (\frac{d}{2} - 1)\omega_0 \phi$, where

¹⁴Below we set $\hbar = 1$ for notational simplicity.

ω_0 is p -independent. The second term in (3.4) can also be made invariant by setting $\delta_{\omega_0} h = 2\omega_0 h + \delta'_{\omega_0} h$, where $\delta'_{\omega_0} h = 2\omega_0 * h - 2\omega_0 h + [\omega_0, h]_* + \delta''_{\omega_0} h$. Here $\delta''_{\omega_0} h$ takes into account the variation of the symbol map under the variation of the metric. More precisely, because the map between the operators and the symbols is one-to-one, one can always trade a variation of the symbol map for an appropriate variation $\delta''_{\omega_0} h$ of h . If we denote by $\mathfrak{R}(g, h)$ the operator associated by the symbol map to the symbol h then

$$\mathfrak{R}(g + 2\omega_0 g, h) = \mathfrak{R}(g, h + \delta''_{\omega_0} h) + O(\omega_0^2). \quad (3.8)$$

One can then represent the variation of h as $\delta_{\omega_0} h = \{\omega_0, h\} + \delta''_{\omega_0} h = 2\omega_0 h + \delta'_{\omega_0} h$. It follows from the structure of the star product and the symbol map that $\delta'_{\omega_0} h_s$ is linear in the CHS fields and only depends on h_{s_i} with $s_i > s$.¹⁵

We conclude that the action (3.4) has an infinitesimal symmetry which is the direct analog of the usual Weyl transformations

$$\delta_{\omega_0} g^{\mu\nu} = 2\omega_0 g^{\mu\nu}, \quad \delta_{\omega_0} h = 2\omega_0 h + \delta'_{\omega_0} h, \quad \delta_{\omega_0} \phi = \left(\frac{d}{2} - 1\right)\omega_0 \phi. \quad (3.9)$$

It will be called the deformed Weyl symmetry in what follows.

4 Covariant expansion of the CHS action in a non-trivial metric

Starting with the covariant version (3.4) of the scalar field action minimally coupled to the CHS fields one can integrate out the scalar ϕ and extract the local log-divergent part $S[g, h]$ of the resulting induced action as in (1.3). This local term is invariant under the h -field part (3.5) of the gauge symmetries (3.5),(3.6) as well as under the symmetries (3.7),(3.9) of the original scalar action and thus provides a natural definition of the CHS action $S[g, h]$ in a general metric background.

4.1 Expansion of the CHS action

Let us specify to the case of $d = 4$ and consider the expansion of $S[g, h]$ in powers of h_s

$$S[g, h] = S[g] + S_1[g, h] + S_2[g, h] + \dots, \quad (4.1)$$

$$S_1[g, h] = \sum_s \int d^4x \sqrt{g} K_{\mu_1 \dots \mu_s}[g] h^{\mu_1 \dots \mu_s}, \quad S_2 = \sum_{s, s'} \int h_s O_{ss'}[g] h_{s'}, \dots \quad (4.2)$$

¹⁵ The only nontrivial point is to check this for $\delta''_{\omega_0} h_s$. The terms in the $\widehat{h}\phi$ involving s derivatives of ϕ have the structure $h^{\mu_1 \dots \mu_t} \partial_{\mu_1} \dots \partial_{\mu_s} \phi$ where contractions of indices between the two groups may be via δ -symbol, $\Gamma_{\nu\rho}^\mu$, and the curvature and its derivatives. It is clear that any such contraction can be nonvanishing only for $t \geq s$ (note the number of upper and lower indices in Γ , R etc.), and, moreover, at $t = s$ one can have the leading contribution where only δ -symbols are employed in the contraction. Furthermore, the variations of the above expression under the change of $g^{\mu\nu}$ and the respective change of the connection, curvature, etc., can be compensated by the variation of h_s . This way one finds the compensating transformation $\delta''_{\omega_0} h_s$ which, by construction, is proportional to h_t with $t > s$.

Here we ignore total derivatives and hence $K_s = (K_{\mu_1 \dots \mu_s})$ can be assumed to be a local function of the metric g . The diffeomorphisms (3.7) transform g and h_s through themselves. Under the deformed Weyl transformations (3.9) g gets rescaled while h_s transforms into h_t with $t \geq s$. As $S[g]$ must be invariant under both diffeomorphisms and the usual Weyl transformations of the metric g and is local, it should be the standard $C_{\mu\nu\lambda\rho}^2$ Weyl action (cf. the discussion of Weyl invariants in Appendix B).

As the diffeomorphism and the deformed Weyl symmetries are homogeneous in h , the linear in h term S_1 must be invariant on its own. Thus $K_{\mu_1 \dots \mu_s}[g]$ should be a tensor under the diffeomorphisms and should vanish (or give a total derivative) if $g_{\mu\nu}$ is flat. The fact that the flat-space CHS action has no terms linear in h is clear directly from (1.2),(1.3). Indeed, as h_s has mass dimension $2 - s$ and the CHS action is local and dimensionless, K_s in S_1 should have dimension $2 + s$, i.e. it should have a structure $\nabla^{2+s} + R\nabla^s + \dots + R^{\frac{2+s}{2}}$ where R is the curvature. We shall ignore the leading highest derivative term as it gives a total derivative in (4.2).

Let us note that for a flat $g_{\mu\nu}$ background the quadratic in h_s term is not manifestly diagonal before one performs the algebraic redefinition of the fields (that takes care of the traces of the fields, i.e. is related to the algebraic part of the gauge transformations [4]). For a non-trivial $g_{\mu\nu}$ one will face a more serious non-diagonality issue due to terms involving the curvature of $g_{\mu\nu}$ that mix fields of different spin; this is related to the differential part of the gauge transformations.

Suppose that $g^{\mu\nu} = g_0^{\mu\nu}(x)$ and $h_s = 0$ (for all s) is a particular solution of the equations corresponding to $S[g, h]$. The necessary and sufficient conditions for that are (ignoring total derivative terms)

$$\left. \frac{\delta S[g]}{\delta g} \right|_{g=g_0} = 0, \quad K_{\mu_1 \dots \mu_s}[g_0] = 0. \quad (4.3)$$

Thus g_0 should be Bach-flat and K_s should vanish on a Bach-flat background. Then the expansion of (4.1) near this solution reads as

$$S[g_0, h] = S[g_0] + S_2[g_0, h] + \dots, \quad (4.4)$$

where we set to zero the perturbation $\bar{h}^{\mu\nu}$ of $g^{\mu\nu}$ itself. As $g^{\mu\nu}$ is not affected by the gauge transformations (3.5),(3.6) the term $S_2[g_0, h]$ should be invariant under the linearized version of (3.5), i.e.

$$\delta h_s = \left([g_0 + \mathcal{R}_0, \epsilon]_* + \{g_0 + \mathcal{R}_0, \omega\}_* \right) \Big|_s. \quad (4.5)$$

Here $A|_s$ denotes the projection to spin s of the generating function $A(x, p)$, i.e. the term of homogeneity s in p_μ .

It follows from the structure of the star-product that one can consistently put to zero all the fields with $s > s_0$ along with ϵ parameters of homogeneity $> s_0 - 1$ and ω of homogeneity $> s_0 - 2$ in p . Hence $S_2[g_0, h]|_{h_{s>s_0}=0}$ is invariant under the linearized gauge transformations (4.5) with ϵ of degree $< s_0$ and ω of degree $< s_0 - 1$.

Let us now show the vanishing of $K_s[g_0]$ in Bach-flat background at least to first order in the background curvature. This will generalize the result of [13] that the $s = 3$ kinetic operator is gauge-invariant in a Bach-flat background to linear order in the curvature.

4.2 Conditions for the vanishing of the linear fluctuation term

Let us study the consequences of the deformed Weyl symmetry (3.9) for the structure of $K_{\mu_1 \dots \mu_s}[g]$. It is useful to introduce the following notation for the transformation of K_s under $\delta g^{\mu\nu} = 2\omega_0 g^{\mu\nu}$:

$$\delta'_{\omega_0} K_{\mu_1 \dots \mu_s} = \delta_{\omega_0} K_{\mu_1 \dots \mu_s} - 2\omega_0 K_{\mu_1 \dots \mu_s} . \quad (4.6)$$

Consider the variation of $\sqrt{g}K_s h_s$ under the deformed Weyl transformation. Since $\delta' h_0$ does not depend on h_0 , the only term proportional to h_0 is $\delta' K_0 h_0$ and hence $\delta' K_0 = 0$. This implies that K_0 is Weyl invariant with weight 2 (i.e. behaves like $g^{\mu\nu}$) but there are no such non-trivial invariants (see Appendix B).

Let us proceed by induction. Assuming that we have shown that $K_r = 0$ for $r < s$, let us consider the variation of $\sqrt{g}K_s h_s$ under the deformed Weyl transformation. Concentrating on the terms in the variation proportional to h_s gives

$$(\delta'_{\omega_0} K_s) h_s + \sum_{l=1}^{\infty} K_l (\delta' h_l) \Big|_s = 0 , \quad (4.7)$$

where $(\delta' h_l) \Big|_s$ denotes the terms in $\delta' h_l$ that are proportional to h_s . Note that $(\delta' h_l) \Big|_s = 0$ for $l \geq s$. Taking this and $K_l = 0$ for $l < s$ into account one finds $\delta'_{\omega_0} K_s = 0$ and hence K_s is a tensor which is Weyl invariant of weight 2.

Next, let us consider the gauge transformations (3.5). The gauge variation of S_1 under the inhomogeneous in h part $([g + \mathcal{R}, \epsilon]_* + \{g + \mathcal{R}, \omega\}_*) \Big|_s$ of (3.5) should vanish. Setting to zero all the fields of spin $> s$ and the associated gauge parameters one gets

$$\int d^4 x \sqrt{g} K_{\mu_1 \dots \mu_s} \nabla^{(\mu_1} \epsilon^{\mu_2 \dots \mu_s)} = 0 , \quad \int d^4 x \sqrt{g} K_{\mu_1 \dots \mu_s} g^{(\mu_1 \mu_2} \omega^{\mu_3 \dots \mu_s)} = 0 . \quad (4.8)$$

This leads to

$$\text{Tr } K_s = 0 , \quad \nabla^{\mu_1} K_{\mu_1 \mu_2 \dots \mu_s} = 0 , \quad (4.9)$$

i.e. K_s should have the same properties as a covariantly-conserved traceless current. The relations (4.8) and (4.9) have direct generalizations to $d > 4$ dimensions.

We have thus shown that under the induction assumption K_s should be a Weyl invariant tensor of weight 2 which is also traceless and covariantly conserved. The totally symmetric traceless tensors of Weyl weight -2 are called in the math literature as “admissible invariants”. For $s \leq 3$ these invariants are known explicitly [22]: for $s = 1$ any invariant vanishes; for $s = 2$ it is proportional to the Bach tensor; for $s = 3$ it is proportional to the Eastwood–Dighton tensor:

$$E_{\mu\nu\rho} = \Psi_{ABCD} \nabla^{DD'} \Psi_{A'B'C'D'} - \Psi_{A'B'C'D'} \nabla^{DD'} \Psi_{ABCD} . \quad (4.10)$$

Here we resorted to the spinor conventions where Ψ_{ABCD} and $\Psi_{A'B'C'D'}$ are Weyl spinors corresponding to (anti)self-dual components of the Weyl tensor.

Considering general s , let us restrict attention to terms in K_s which are linear in the curvature or Weyl tensor C . Then we will have $\nabla^s C$ like terms where s indices are symmetrised and 4 indices are contracted by $g^{\mu\nu}$. Since $\nabla^s C$ should be a totally symmetric tensor and since C has 4 indices and is traceless, two of the derivatives should act on C itself. Such terms should vanish on a Bach-flat background.

It would be important to extend the above argument of the vanishing of K_s beyond the linear in curvature terms. Let us make few comments that may be useful for an attempt to prove this. As the metric g should actually be a background for the spin 2 field h_2 , there should be a hidden gauge symmetry which transforms g and h_2 in such a way that their sum $g + h_2$ remains invariant while all other fields h_s also transform to compensate for the change of the symbol map. This symmetry may be useful to eliminate some unwanted terms. Another remark is that we are dealing with the CHS theory involving an infinite set of fields but so far made use of only some of the gauge symmetries that preserve the subspace of field configurations where only a finite collection of fields are non-vanishing. In particular, for the above arguments to work it is enough to compute the CHS action as the divergent part of the scalar effective action (1.3) with $h_r = 0$ for $r > s$. This way one may avoid subtleties related to the fact that the full space of CHS fields is infinite-dimensional. Finally, let us mention that the entire construction can probably be made more geometrical by employing the conformally equivariant quantization which is known [23, 24] for generic conformal manifolds.

4.3 Gauge invariance of spin- s quadratic term to first order in curvature

As we have argued above, $K_s[g_0]$ must vanish at least up to terms of second order in the curvature¹⁶ if the background metric g_0 is Bach-flat. The gauge invariance of the complete action (4.1) at the zeroth and the first order in h gives

$$\int \sqrt{g} K_s \delta_\epsilon^0 h_s = 0, \quad \int \sqrt{g} K_s \delta_\epsilon^1 h_s + \int \sqrt{g} \frac{\delta S_2}{\delta h_s} \delta_\epsilon^0 h_s = 0, \quad (4.11)$$

where δ^0 and δ^1 denote the leading and the linear in h parts of the gauge transformation. As $K_s \sim C^2$ (here C denotes the Weyl tensor and its Weyl-covariant derivatives, see Appendix B) the second equality implies

$$\delta_{\epsilon, \omega}^0 S_2[g_0, h] = O(R^2), \quad (4.12)$$

where $\delta_{\epsilon, \omega}^0$ denotes the gauge transformation linearized around $g = g_0$, $h_s = 0$. To zeroth order in the curvatures the gauge transformations are explicitly

$$\delta_\epsilon h_s = (p_a \nabla^a) \epsilon_{s-1}, \quad \delta_\omega h_s = -\frac{1}{2} g^{\mu\nu} p_\mu p_\nu \omega_{s-2} + \dots, \quad (4.13)$$

¹⁶By curvature terms we always mean the products of Riemann tensor and its covariant derivatives.

where dots denote terms involving $(\square)^l \omega_{s-2+2l}$ for $l = 1, 2, \dots$

If we now set to zero all the fields with $s > s_0$ and their associated gauge parameters, then for $s = s_0$ (4.13) gives the exact linearized gauge transformation. Indeed, the implicit terms in the expression for $\delta_\omega h_s$ are not present for $s = s_0$ while the curvature contributions may only affect fields with the spins lower than s_0 . Moreover, to zeroth order in the curvature the implicit terms in (4.13), which are present for $s < s_0$, can be removed [4] by the field and the gauge parameter redefinition of the following form

$$h'_s = h_s + Y_s(h_{s+2}, \dots, h_{s_0}), \quad \epsilon'_s = \epsilon_s + Z_s(\epsilon_{s+2}, \omega_{s+2}, \dots, \epsilon_{s_0}, w_{s_0}). \quad (4.14)$$

Upon this redefinition the gauge transformation takes the standard diagonal form, with the usual derivatives replaced by the covariant ones. This implies that to zeroth order in the curvature the term $S_2[g_0, h]$ is just a direct sum of the standard quadratic actions for all spins $1, 2, \dots, s_0$.

Let us now include terms of first order in the curvature. Because to zeroth order in curvature $S_2[g_0, h]$ is diagonal (does not contain terms mixing different spins) the gauge invariance implies that the quadratic in h_{s_0} term in $S_2[g_0, h]$ is gauge invariant on its own up to terms of second order in curvature. This generalizes the spin 3 statement from [13] to any integer spin case.

Let us note that, strictly speaking, in the above considerations we made use of the expansion in Riemann curvature while the vanishing of K_s was shown to first order in the Weyl curvature. This is the same for special case of Ricci-flat backgrounds, but there are Bach-flat backgrounds that are not Ricci-flat. Taking into account the (deformed) Weyl invariance it should be possible to demonstrate also the gauge invariance to first order in the Weyl curvature.

4.4 Spin 3 example

Let us now assume that the background metric is chosen such that both the Bach tensor (1.4) and the Eastwood-Dighton tensor (4.10) vanish, i.e. $B_{\mu\nu} = 0 = E_{\mu\nu\rho}$. For an algebraically-general Weyl tensor this implies that the metric is conformally Einstein [25, 22]. Unfortunately, the vanishing of K_3 in (4.2),(4.9) does not directly imply that the spin 3 CHS field kinetic term is always consistent (i.e. gauge-invariant) on such a background. Taking $\epsilon = \epsilon^{ab} p_a p_b$ and extracting the linear in h_3 contribution in the second equation in (4.11) one gets:

$$\int d^4x \sqrt{g} K_4[h_3, \epsilon_2]_* \Big|_4 + \int d^4x \sqrt{g} h_3 \frac{\delta^2 S_2}{\delta h_3 \delta h_1} \delta_{\epsilon_2}^0 h_1 + \int d^4x \sqrt{g} h_3 \frac{\delta^2 S_2}{\delta h_3 \delta h_3} \delta_{\epsilon_2}^0 h_3 = 0. \quad (4.15)$$

Thus if K_4 is nonvanishing, our argument does not in general imply that the spin 1 plus spin 3 system is consistent.

It is clear from the structure of (4.15) that on a non-trivial background the spin 3 field may mix with the spin 1 in the quadratic term S_2 in (4.1). To understand the reason for this mixing let us go back to the discussion in sections 2 and 3 and consider the linearized gauge transformations

around the vacuum Hamiltonian $H_0 = -\frac{1}{2}g^{\mu\nu}p_\mu p_\nu = -\frac{1}{2}\eta^{ab}p_a p_b$ in (2.4). As was noted above, it follows from the structure of the star-product that the linearized gauge transformations with parameters ϵ and ω of degree $s-1$ and $s-2$ respectively can only affect the fields of spins $s, s-2, s-4, \dots$. Thus the simplest nontrivial system is that of spins 1 and 3. For $s=1$ field the gauge transformations are standard. For $s=3$ the parameters are $\epsilon^{ab}p_a p_b$ and $\omega^a p_a$. Let us first consider the gradient-like transformation

$$\delta(h^{abc}p_a p_b p_c) = \left[-\frac{1}{2}\eta^{ab}p_a p_b, \epsilon^{cd}p_c p_d\right]_* \Big|_3 = p_a p_b p_c (\nabla^a \epsilon^{bc}), \quad (4.16)$$

where we projected to the spin 3 component. This is thus a natural covariantisation of the flat-space gradient gauge transformation. However, the transformation generated by $\epsilon^{cd}p_c p_d$ gives also a non-zero contribution to the spin 1 sector:¹⁷

$$\delta(h^a p_a) = \left[-\frac{1}{2}\eta^{ab}p_a p_b, \epsilon^{cd}p_c p_d\right]_* \Big|_1 = -\frac{4}{3}R_{bcd}^a \nabla_a \epsilon^{cd} p^b. \quad (4.17)$$

Thus the linearized gauge transformations with parameters ϵ^{ab} and ω^a will act on spin 1 field as

$$\delta h_a = R_{ab}{}^c{}_d \nabla_c \epsilon^{bd} - \nabla^2 \omega_a. \quad (4.18)$$

While the second term here can be removed by a field redefinition (the “dressing map” of [4]) $h_a \rightarrow h_a + c \nabla^2 h_{ab}{}^b$, the first term is non-trivial. Its presence implies that the standard Maxwell $\partial h_1 \partial h_1$ term in the quadratic action S_2 can not be invariant under such transformation. As a result, we should then have $h_1 h_3$ mixing, i.e. should add non-diagonal $R \nabla h_1 \nabla h_3 + R R h_1 h_3$ terms to compensate for the variation of h_1 quadratic term under the h_3 gauge transformation in (4.18).¹⁸

As we have seen above, to first order in the curvature these mixing terms do not affect the gauge invariance of the quadratic term S_2 under the transformation with parameters ϵ_{s-1} and ω_{s-2} . However, to second order in the curvature the mixing terms can not be neglected. Then it is natural to expect that in general only a system of all spins $s, s-2, s-4, \dots$ can be well-defined on a sufficiently curved background. The presence of non-diagonal terms in S_2 on curved background is thus expected in general and deserves further study.¹⁹

¹⁷Let us stress that the above relations are complete as terms of degree higher than 2 in the covariantly constant lifts of η and ϵ can not contribute. For comparison, in the sector of spin 2 and spin 0 fields with $H = \eta^{ab}p_a p_b + h_0(x)$, the gauge transformations with parameters $\epsilon = \epsilon^a(x)p_a$ and $\omega = \omega(x)$ read (restricted to h_0 , cf. also [4])

$$\delta h_0 = \epsilon^a \nabla_a h_0 + 2\omega h_0 + \frac{1}{2}\eta^{ab} \nabla_a \nabla_b \omega.$$

Using the transformation law of the scalar curvature $\delta_\omega R = 2\omega R + \eta^{ab} \nabla_a \nabla_b \omega$ under Weyl transformations of the metric $g \rightarrow \omega g$ one finds that $h'_0 = h_0 - \gamma R$ transforms homogeneously: $\delta h'_0 = \epsilon^a \nabla_a h'_0 + 2\omega h'_0$. It follows from the above transformation law that one can consistently put h'_0 to zero in the scalar field action $\int d^d x \sqrt{g} (\phi^* \rho (\tilde{\eta} + \tilde{h}_0) \phi) \Big|_{y=0} = \int d^d x \sqrt{g} \phi^* (-\nabla^2 + \gamma R + h'_0) \phi$.

¹⁸Note that in the constant curvature space where $R_{abcd} = \lambda(\eta_{ac}\eta_{bd} - \eta_{bc}\eta_{ad})$ the first term in (4.18) takes the form $\delta h_a = \lambda(\nabla_a \epsilon_b^b - \nabla_b \epsilon_a^b) \sim \nabla_a \epsilon_b^b + \eta^{bc}(\nabla_a \epsilon_{bc})$ and hence can be removed by a combination of field redefinition and gauge parameter redefinition. The same should be true also for conformally-flat metrics.

¹⁹Their presence should not, however, change the computation of conformal anomalies of CHS fields on Ricci-flat

5 Conclusions

In this paper we addressed the question of covariant description of conformal higher spin fields in a non-trivial background. The standard definition of the CHS action (1.3) gives an expansion near flat space and thus is not generally covariant. Given that the spin 2 CHS field should have a natural interpretation of a conformal graviton, one expects that there should be a possibility to rewrite this action in a manifestly covariant form with the spin 2 part represented by the non-linear Weyl action.²⁰

We suggested a way to define the CHS action in a covariant way by using the background metric to define the star product in the associated particle dynamics and thus in the definition of the gauge transformations.

As is well known, the quadratic term in an action expanded near its classical solution should have linearized gauge invariance. For example, the quadratic 4-derivative operator in the Weyl action expanded near Bach-flat background is consistent, i.e. has the standard reparametrization invariance (which is fixed by a background gauge in quantum computations). The same was previously found to be true to linear order in the curvature expansion for the conformal spin 3 operator in a Bach-flat metric [13]. Here we generalized this fact to any conformal higher spin field and commented on a possibility of extending this claim to terms quadratic in the curvature. We also pointed out the presence of curvature-dependent mixing terms in the quadratic part of the conformal higher spin action expanded in a non-trivial background.

Acknowledgements

We would like to thank R. Metsaev, E. Skvortsov and M. Taronna for discussions of related questions. M.G. also wishes to thank N. Boulanger for a useful discussion of Weyl invariants. This work was supported by the Russian Science Foundation grant 14-42-00047. The work of AAT was also supported by the ERC Advanced grant No.290456 and the STFC Consolidated grant ST/L00044X/1.

background in [9] as their contributions will involve derivative ∇R structures that should be absent in 4d conformal anomaly in dimensional grounds.

²⁰It should be noted that a possibility to rewrite the action for an infinite set of fields in a manifestly covariant and local way is not a priori obvious. For a somewhat related discussion in the string theory context see [26].

A Covariant quantization in Fedosov-type approach: quantum version of normal coordinate expansion

Let us recall how to perform quantization on the cotangent bundle in generic coordinates. Let x^μ be coordinates on the base manifold and p_μ their conjugate momenta. The canonical Poisson bracket reads as $\{x^\mu, p_\nu\} = \delta^\mu_\nu$. We would like to define quantization compatible with a given Riemannian metric $g_{\mu\nu}(x)$. Let us introduce frame field e^a_μ and Lorentz connection $\omega^b_{\mu a}$ such that ²¹

$$\nabla e^a = 0, \quad \omega^b_a \eta_{bc} + \omega^b_c \eta_{ba} = 0, \quad g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}. \quad (\text{A.1})$$

In what follows we will use the coordinates $x^\mu, p_a = e^a_\mu p_\mu$ on the cotangent bundle.

Let us introduce extra variables y^a and the star product

$$\circ = \exp \left[\frac{\hbar}{2} \left(\overleftarrow{\frac{\partial}{\partial y^a}} \frac{\partial}{\partial p_b} - \frac{\partial}{\partial p_a} \overleftarrow{\frac{\partial}{\partial y^a}} \right) \right]. \quad (\text{A.2})$$

Proposition A.1. *Given e, ω there exist a nonlinear connection whose covariant derivative (acting on forms with values in functions of y, p) has the form*

$$D = dx^\mu \frac{\partial}{\partial x^\mu} + \hbar^{-1} [e^a p_a + \omega^b_{\mu a} y^a p_b, \cdot]_\circ + \hbar^{-1} [\mathbf{r}, \cdot]_\circ, \quad \mathbf{r} = y^a y^b dx^\mu \mathbf{r}_{\mu ab}(x, y, p) \quad (\text{A.3})$$

and for any $f(x, y, p)$ satisfies

$$DDf(x, y, p) = 0. \quad (\text{A.4})$$

Under the extra condition $e^a_\mu y^a \frac{\partial}{\partial (dx^\mu)} \mathbf{r} = 0$ this connection is unique and is such that \mathbf{r} is linear in p_a .

Proof. The proof is based on using suitable degree of homogeneity in y plus \hbar and acyclicity of the differential $\delta = dx^\mu e^a_\mu \frac{\partial}{\partial y^a}$ in nonzero form-degree. \square

Note that D clearly differentiates \circ -product. If \mathbf{r} is linear in p_a it also satisfies the Poisson bracket version of the flatness condition (i.e. coincides with its classical limit). In what follows we assume that D is minimal (satisfies $e^a_\mu y^a \frac{\partial}{\partial (dx^\mu)} \mathbf{r} = 0$). Terms of degree 4 and less read explicitly as:²²

$$\begin{aligned} \mathbf{r} = dx^\mu \Big[& -\frac{1}{3} R^a_{\mu cb} p_a y^c y^b + (\dots) \nabla_d R^a_{\mu cb} p_a y^c y^b y^d + (\dots) \nabla_e \nabla_d R^a_{\mu cb} p_a y^c y^b y^d y^e + \dots \Big] \\ & + dx^\mu (R^c_{\mu de} R^a_{fbc} p_a y^f y^b y^d y^e + \dots) + \dots \end{aligned} \quad (\text{A.5})$$

²¹We use convention $\nabla(T^a p_a) = dx^\mu \omega^a_{\mu b} T^b p_a, R^a_{\mu\nu b} = \partial_\mu \omega^a_{\nu b} + \omega^a_{\mu c} \omega^c_{\nu b} - (\mu \rightleftharpoons \nu)$. In particular, $\nabla^2 f(x, y, p) = [\frac{1}{2} dx^\mu dx^\nu R^a_{\mu\nu b} y^b p_a, f(x, y, p)]_*$.

²²We shall denote by (\dots) some numerical coefficients precise values of which is not relevant for our discussion.

Here the first line contain terms linear in curvature and its covariant derivatives.²³

Proposition A.2. *For any $f(x, p)$ there exist a unique $\check{f}(x, y, p)$ such that*

$$D\check{f} = 0, \quad \check{f}|_{y=0} = f. \quad (\text{A.6})$$

Moreover, if D is a unique connection such that $e_a^\mu y^a \frac{\partial}{\partial(dx^\mu)} \mathbf{r} = 0$ then for $f(x, -p) = \pm f(x, p)$ the associated \check{f} also satisfies $\check{f}(x, -p) = \pm \check{f}(x, p)$. More precisely, for f of homogeneity s in p_a , \check{f} contains terms of homogeneity $s, s-2, s-4, \dots$

Proof. \check{f} is constructed iteratively in the degree of homogeneity in y and \hbar . For D special \mathbf{r} is linear in p_a so that the star commutator may only reduce the homogeneity in p_a by an even number. \square

It follows that the space of all functions in x, p is isomorphic to covariantly constant functions depending in addition on y -variables. Later on we will need the following example: if $\eta = \frac{1}{2}\eta^{ab}p_ap_b$ then

$$\check{\eta} = \frac{1}{2}\eta^{ab}p_ap_b + \frac{1}{6}R_{bcd}^a p_a p^b y^c y^d + (\text{terms of degree } > 2) \quad (\text{A.7})$$

This is related to the expansion in normal coordinates if one identifies y^a as normal coordinates around x^μ . Note that, in general, terms independent of momenta may appear but they are of order \hbar^2 . For a general element $f = f^{ab}(x)p_ap_b$ quadratic in p_a one has

$$\begin{aligned} \check{f} = f^{ab}(x)p_ap_b + y^a \nabla_a f^{bc} p_b p_c + \frac{1}{2} y^a y^b \nabla_a \nabla_b f^{cd} p_c p_d + (\dots) R_{bcd}^a f^{be} y^c y^d p_a p_e + \\ + \frac{2}{3} \hbar^2 R_{bcd}^a \nabla_a f^{cd} y^b + \dots \end{aligned} \quad (\text{A.8})$$

where dots denote terms of total degree higher than 2. For a linear one

$$\check{f} = f^a p_a + y^b \nabla_b f^a p_a + \frac{1}{2} y^a y^b \nabla_a \nabla_b f^c p_c + (\dots) R_{bcd}^a f^b y^c y^d p_a + (\text{deg} \geq 3 \text{ terms}) \quad (\text{A.9})$$

The above construction gives the covariant $*$ product on contangent bundle: using the above propositions we may define

$$f * g := (\check{f} \circ \check{g})|_{y=0}. \quad (\text{A.10})$$

The consistency of this definition follows from the fact that for any \check{f}, \check{g} satisfying $D\check{f} = D\check{g} = 0$ one has $D(\check{f} \circ \check{g}) = 0$. The above construction of the star product is a version of that of [27] which in turn has its origin in the Fedosov quantization [28].

²³Note that expansion in homogeneity in curvatures $\mathbf{r} = \sum_{i=1}^{\infty} \mathbf{r}_i$ is well defined and the flatness condition decomposes as

$$\nabla \mathbf{r}_i - \delta \mathbf{r}_i + \frac{1}{2} \sum_{l+k=i, l, k \geq 0} [\mathbf{r}_l, \mathbf{r}_k]_{\circ} = 0$$

As an example let us compute explicitly the transformation of the spin 1 under the transformation generated by $\epsilon^{ab} p_a p_b$:

$$\delta(h^a p_a) = [-\frac{1}{2}\eta^{ab} p_a p_b, \epsilon^{cd} p_c p_d]_*|_1 = -\frac{4}{3} R_{bcd}^a \nabla_a \epsilon^{cd} p^b. \quad (\text{A.11})$$

Next, let us describe the representation space in a covariant way. Let ρ denote a map that sends Weyl symbol $f(y, p)$ into the respective operator in coordinate representation (i.e. on functions of y). For instance, $\rho(y_a p_b) = \frac{1}{2}\hbar(y_a \frac{\partial}{\partial y_b} + \frac{\partial}{\partial y_b} y_a)$.

Proposition A.3. *For any wave function $\phi(x)$ there exist a unique lift $\check{\phi}(x, y)$ satisfying $\hbar[\nabla + \rho(e^a p_a + \mathbf{r})]\check{\phi} = 0$ and $\phi|_{y=0} = \phi$.*

To illustrate this, let us explicitly evaluate the lift up to terms of degree 3:

$$\check{\phi} = \phi + y^a \nabla_a \phi + \frac{1}{2} y^a y^b [\nabla_a \nabla_b + (\dots)\hbar^2 R_{ab}]\phi + \dots \quad (\text{A.12})$$

The action of the operator $\widehat{f}(x, \frac{\partial}{\partial x})$ with symbol $f(x, p)$ on the wave function $\phi(x)$ is defined by

$$\widehat{f}\phi = (\rho(\check{f})\Phi)|_{y=0}. \quad (\text{A.13})$$

Note that by construction 1 acts as an identity operator and $\widehat{(f * g)} = \widehat{f} \widehat{g}$ (because ρ is a representation map). This way we have constructed a covariant symbol map that sends functions of x, p to differential operators on x . Note that the map is solely expressed in terms of covariant derivatives, frame field, and curvature (along with its covariant derivatives). This shows that although the map is written in terms of local coordinates and local frame it does not depend on the choice of coordinates and the frame.

The above technique allows to reformulate the relations (2.3) and (2.7) in manifestly coordinate-independent terms. In so doing the component fields entering $H(x, p)$ transform as tensors under a change of coordinates. By a suitable field redefinition one can also achieve that they transform homogeneously under the linearized gauge transformations (see the end of this appendix for spin 2 case).

Let us now discuss the inner product. The minimal choice is

$$\langle \phi, \chi \rangle = \int d^d x \sqrt{g} \phi^*(x) \chi(x) \quad (\text{A.14})$$

The question is how to identify (anti)hermitian operators at the level of symbols.

Proposition A.4. *Real (imaginary) symbols correspond to hermitian (antihermitian) operators.*

Proof. First of all we show that for $f(x, p)$ real (imaginary) the respective lift $\check{f}(x, p, y)$ is also real (imaginary). Let us for definiteness consider real f . It is enough to assume all coefficients to be real so that $f(x, p)$ contains only even powers of p_a . By inspecting the recursive construction

of \check{f} we see that odd powers of p can not appear as well as imaginary coefficients (we assume that the metric, frame field and connection are real). Finally, because \check{f} is real $\rho(\check{f})$ is formally hermitian in when represented on wave functions of y^a where the conjugation rules are $(y^a)^\dagger = y^a$ and $\frac{\partial}{\partial y^a}^\dagger = -\frac{\partial}{\partial y^a}$ (in this case dependence on x^μ is irrelevant and as before $x^{\mu\dagger} = x^\mu$).

It is enough to check this statement for operators whose symbols are of zeroth and first order in p . Indeed, such operators generate the entire algebra. For $f = f(x)$ the statement is obvious. For $f = v^a(x)p_a$ we have (this is just a rewriting of (A.9))

$$\check{f}(x, p, y) = \tilde{v}^a(x, y)p_a, \quad \tilde{v}^a = v^a + y^b \nabla_b v^a + O(y^2) \quad (\text{A.15})$$

Because $\rho(f)$ is formally antihermitian on wave functions of y we have

$$\begin{aligned} \int d^d x \sqrt{g} \phi^* \hat{f} \chi &= \int d^d x \sqrt{g} (\check{\phi}^* \rho(\check{f}) \check{\chi})|_{y=0} \\ &= \frac{1}{2} \int d^d x \sqrt{g} \left(\check{\phi}^* (\tilde{v}^a \frac{\partial}{\partial y^a} + \frac{\partial}{\partial y^a} \tilde{v}^a) \check{\chi} \right)|_{y=0} \\ &= - \int d^d x \sqrt{g} ((\rho(\check{f}) \check{\phi})^* \check{\chi})|_{y=0} + \int d^d x \sqrt{g} \left(\frac{\partial}{\partial y^a} (\check{\phi}^* \tilde{v}^a \check{\chi}) \right)|_{y=0}. \end{aligned} \quad (\text{A.16})$$

Using that $(\frac{\partial}{\partial y^a} X)|_{y=0} = \nabla_a Y$ for some Y , where X is ϕ^* or χ or \tilde{v}^a , the integrand of the last term can be rewritten as $\sqrt{g} \nabla_a X^a = \sqrt{g} \nabla_\mu X^\mu = \partial_\mu (\dots)^\mu$ and hence the integral vanishes under the standard assumptions. \square

To summarize, we have constructed a covariant (independent of the choice of local coordinates) description of quantum mechanics on the cotangent bundle. We thus have all the required ingredients: representation space, inner product, operators, symbols and symbol-map.

Now we are able to write a covariant version of the action (2.6) by taking

$$S[e, \omega, h] = \int d^d x \sqrt{g} \phi^* [\hat{p}^2 + \widehat{h(x, p)}] \phi = \int d^d x \sqrt{g} \check{\phi}^* \rho[\check{p}^2 + h(\check{x}, p)] \check{\phi}. \quad (\text{A.17})$$

Note that the action depend on the frame field only through $g_{\mu\nu}$ and hence using $p_\mu = e_\mu^a p_a$ instead of p_a one can treat this as an action depending on ϕ, g, h_s as we do in the main text.²⁴

B Weyl invariants

Let us briefly recall some known results on the structure of the conformal and diffeomorphism invariants. More precisely, we are interested in (tensor valued) local functions $K_{\mu_1 \dots \mu_s}$ of the metric and its derivatives (cf. (4.2)) that transform covariantly under the diffeomorphisms and Weyl transformations. It turns out that a candidate invariant is a polynomial in

$$g_{\mu\nu}, \quad C_{\mu\nu\rho\sigma}, \quad \mathcal{D}_\alpha C_{\mu\nu\rho\sigma}, \quad \mathcal{D}_\alpha \mathcal{D}_\beta C_{\mu\nu\rho\sigma}, \quad \dots \quad (\text{B.1})$$

²⁴If we kept using frame field and Lorentz connection we would have in addition local Lorentz symmetry (as all fiber indices are contracted properly and only Lorentz covariant derivatives are entering all the expressions).

with indices properly contracted by $g^{\mu\nu}$. Here $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and \mathcal{D}_α denotes a Weyl-covariant derivative related to the so-called Thomas D-derivative.²⁵ In general, such polynomial is not invariant under Weyl transformations as in contrast to $g^{\mu\nu}$ and $C_{\mu\nu\rho\sigma}$ the transformation law of $\mathcal{D}_\alpha \dots \mathcal{D}_\beta C_{\mu\nu\rho\sigma}$ may involve a gradient of the Weyl parameter ω_0 . Hence the invariance condition imposes extra constraints on the structure of the polynomial. For more details we refer to [29] and references there.

Let us analyze the necessary condition for a rank s tensor-valued local function to be diffeomorphism covariant and Weyl covariant with weight w . Taking into account that $g^{\mu\nu}$ has Weyl weight 2 while $g_{\mu\nu}$ and $\mathcal{D}_\mu \dots \mathcal{D}_\nu C_{\alpha\beta\gamma\delta}$ have weight -2 , we get

$$2n_g + 4n_C + n_{\mathcal{D}} - 2n^g = s, \quad -2n_g - 2n_C + 2n^g = w, \quad (\text{B.2})$$

where n_g , n_C , $n_{\mathcal{D}}$ and n^g denote, respectively, the numbers of $g_{\mu\nu}$, C , \mathcal{D} and $g^{\mu\nu}$ factors in a polynomial. The first equation counts indices while the second counts the Weyl weight. As a consequence, we have

$$2n_C + n_{\mathcal{D}} = w + s. \quad (\text{B.3})$$

Consider, for example, a scalar invariant which is an integrand of $S_0[g] = \int d^4x \sqrt{g} L_0$. One finds that L_0 is Weyl invariant of weight 4 for which (B.3) has two solutions $n_C = 1, n_{\mathcal{D}} = 2$ and $n_C = 2, n_{\mathcal{D}} = 0$. The first one gives zero (as C is traceless) so one ends up with $L_0 = C^2$, i.e. the well-known Weyl gravity Lagrangian.

Next, let us consider a rank-one tensor K_μ of weight $w = 2$ appearing in (4.2). Then we have only one nontrivial solution: $n_C = 1, n_{\mathcal{D}} = 1$. It should again vanish as here at least two indices of the Weyl tensor should be contracted with the metric.

For a polynomial $K_{\mu\nu}$ with $s = 2, w = 2$ we have two solutions: $n_C = 2, n_{\mathcal{D}} = 0$ and $n_C = 1, n_{\mathcal{D}} = 2$. The latter one necessarily contains two derivatives contracted with the indices of C and hence should vanish on a Bach-flat background. The former can be brought to the following form

$$k_1 g_{\mu\nu} C_{\alpha\beta\gamma\rho} C^{\alpha\beta\gamma\rho} + k_2 C_{\mu\alpha\beta\gamma} C^{\alpha\beta\gamma}_{\nu}. \quad (\text{B.4})$$

Imposing the tracelessness ($k_1 = -\frac{1}{4}k_2$) and covariant conservation conditions (4.8) this can be shown to vanish on a Ricci-flat background using $\nabla_{[\mu} C_{\nu\alpha]\beta\gamma} = 0$; Weyl-covariance implies that same should be true on a Bach-flat background.

The analysis for $s > 2$ becomes rather involved. Considerable simplification can be achieved by employing the spinor formalism in $4d$. In this approach the self-dual (anti-self-dual) component of C is represented by the totally symmetric spinor Ψ_{ABCD} ($\Psi_{A'B'C'D'}$) where $A = 1, 2$ ($A' = 1, 2$). The invariant contractions of indices are performed with the help of the antisymmetric tensor ϵ^{AB} or $\epsilon^{A'B'}$. In particular, the Minkowski metric $\eta_{\mu\nu}$ in spinorial notations reads as $\eta_{AA', BB'} = 2\epsilon_{AB}\epsilon_{A'B'}$ (for a concise exposition see, e.g., [30] and refs. therein).

²⁵The first \mathcal{D} -derivative of Weyl tensor is the same as the ordinary covariant derivative, i.e. $\mathcal{D}_\alpha C_{\beta\gamma\delta\rho} = \nabla_\alpha C_{\beta\gamma\delta\rho}$.

For example, for $s = 2$, by writing the spinorial counterpart of (B.4) one finds that the second term necessarily vanishes so that the tracelessness condition implies that the first term vanishes as well.

For $s = 3$ we have $K_{\mu\nu\rho}$ which, according to (B.3), can not have terms of order higher than 2 in Weyl tensor and its Weyl-covariant derivatives. As the linear in $C_{\mu\nu\rho\sigma}$ term vanishes on Bach-flat background let us concentrate on the quadratic contribution which should involve only one covariant derivative. In the spinorial approach $K_{\mu\nu\rho}$ is described by $K_{AA'BB'CC'}$ to which only the following terms may contribute

$$\Psi_{ABCD}\nabla_{EE'}\Psi_{A'B'C'D'}, \quad \Psi_{A'B'C'D'}\nabla_{EE'}\Psi_{ABCD}, \quad (\text{B.5})$$

where the indices are contracted with the ϵ -tensors. It is clear that there is only one inequivalent contraction that leaves $3 + 3$ free indices. It results in the following general expression:

$$n_1 \Psi_{ABCD}\nabla^{DD'}\Psi_{A'B'C'D'} + n_2 \Psi_{A'B'C'D'}\nabla^{DD'}\Psi_{ABCD}. \quad (\text{B.6})$$

The Weyl covariance of $K_{\mu\nu\rho}$ implies that $n_2 = -n_1$ in which case the above expression is proportional to the Eastwood-Dighton tensor $E_{\mu\nu\rho}$ in (4.10). It is known to vanish for the metric conformal to the Einstein one. Note that the Eastwood-Dighton tensor is by construction trace-free and its divergence is proportional to the Bach tensor and thus vanishes on a Bach-flat background.

Finally, let us list two useful relations in spinorial notations. The Bianchi identity for the Weyl tensor reads

$$\nabla_{B'}^A \Psi_{ABCD} = \nabla_B^{A'} \Phi_{CDA'B'} - 2\epsilon_{B(C} \nabla_{D)B'} \Lambda, \quad (\text{B.7})$$

where $\Phi_{ABA'B'}$ is the trace-free Ricci spinor and Λ is a multiple of the scalar curvature. The Bach tensor is given by

$$B_{AA'BB'} = 2(\nabla_{A'}^C \nabla_{B'}^D + \Phi^{CD}_{A'B'})\Psi_{ABCD} = 2(\nabla^{C'}_A \nabla^{D'}_B + \Phi^{C'D'}_{AB})\Psi_{A'B'C'D'}. \quad (\text{B.8})$$

References

- [1] E. S. Fradkin and A. A. Tseytlin, “Conformal supergravity,” Phys.Rept. **119** (1985) 233–362.
- [2] E. S. Fradkin and V. Y. Linetsky, “Cubic Interaction in Conformal Theory of Integer Higher Spin Fields in Four-dimensional Space-time,” Phys.Lett. **B231** (1989) 97.
- [3] A. A. Tseytlin, “On limits of superstring in $AdS_5 \times S^5$,” Theor.Math.Phys. **133** (2002) 1376–1389, hep-th/0201112.
- [4] A. Y. Segal, “Conformal higher spin theory,” Nucl.Phys. **B664** (2003) 59–130, hep-th/0207212.
- [5] X. Bekaert, E. Joung, and J. Mourad, “Effective action in a higher-spin background,” JHEP **1102** (2011) 048, 1012.2103.
- [6] S. Giombi, I. R. Klebanov, S. S. Pufu, B. R. Safdi, and G. Tarnopolsky, “AdS Description of Induced Higher-Spin Gauge Theory,” JHEP **1310** (2013) 016, 1306.5242.
- [7] E. Joung, S. Nakach, and A. A. Tseytlin, “Scalar scattering via conformal higher spin exchange,” JHEP **02** (2016) 125, 1512.08896.
- [8] M. Beccaria, S. Nakach, and A. A. Tseytlin, “On triviality of S-matrix in conformal higher spin theory,” JHEP **09** (2016) 034, 1607.06379.
- [9] A. A. Tseytlin, “On partition function and Weyl anomaly of conformal higher spin fields,” Nucl.Phys. **B877** (2013) 598–631, 1309.0785.
- [10] S. Giombi, I. R. Klebanov, and B. R. Safdi, “Higher Spin AdS_{d+1}/CFT_d at One Loop,” Phys.Rev. **D89** (2014) 084004, 1401.0825.
- [11] M. Beccaria and A. Tseytlin, “On higher spin partition functions,” J.Phys. **A48** (2015), no. 27, 275401, 1503.08143.
- [12] M. Beccaria and A. A. Tseytlin, “Higher spins in AdS_5 at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT ,” JHEP **1411** (2014) 114, 1410.3273.
- [13] T. Nutma and M. Taronna, “On conformal higher spin wave operators,” JHEP **1406** (2014) 066, 1404.7452.
- [14] M. Beccaria, X. Bekaert, and A. A. Tseytlin, “Partition function of free conformal higher spin theory,” JHEP **1408** (2014) 113, 1406.3542.
- [15] R. R. Metsaev, “Ordinary-derivative formulation of conformal totally symmetric arbitrary spin bosonic fields,” JHEP **1206** (2012) 062, 0709.4392.

- [16] R. R. Metsaev, “Arbitrary spin conformal fields in (A)dS,” Nucl. Phys. **B885** (2014) 734–771, 1404.3712.
- [17] E. S. Fradkin and A. A. Tseytlin, “Renormalizable asymptotically free quantum theory of gravity,” Nucl. Phys. **B201** (1982) 469–491.
- [18] A. Y. Segal, “Point particle in general background fields vs. free gauge theories of traceless symmetric tensors,” Int. J. Mod. Phys. **A18** (2003) 4999–5021, hep-th/0110056.
- [19] M. Grigoriev, “Off-shell gauge fields from BRST quantization,” hep-th/0605089.
- [20] X. Bekaert and M. Grigoriev, “Higher order singletons, partially massless fields and their boundary values in the ambient approach,” Nucl.Phys. **B876** (2013) 667–714, 1305.0162.
- [21] G. T. Horowitz, J. D. Lykken, R. Rohm, and A. Strominger, “A purely cubic action for string field theory,” Phys.Rev.Lett. **57** (1986) 283–286.
- [22] C. LeBrun, “Twistors, Ambitwistors, and Conformal Gravity,” in Twistors in Mathematics and Physics, T. N. Bailey and R. J. Baston, eds., pp. 71–86. Cambridge University Press, 1990. Cambridge Books Online.
- [23] F. Radoux, “An Explicit Formula for the Natural and Conformally Invariant Quantization,” Letters in Mathematical Physics (Sept., 2009) 0902.1543.
- [24] J. Silhan, “Conformally invariant quantization – towards complete classification,” ArXiv e-prints (Mar., 2009) 0903.4798.
- [25] C. N. Kozameh, E. T. Newman, and K. P. Tod, “Conformal Einstein spaces,” General Relativity and Gravitation **17** (1985), no. 4, 343–352.
- [26] A. A. Tseytlin, “String Field Theory in Components: General Covariance Versus Massive Fields,” Phys. Lett. **B185** (1987) 59–64.
- [27] M. Bordemann, N. Neumaier, and S. Waldmann, “Homogeneous Fedosov Star Products on Cotangent Bundles II: GNS Representations, the WKB Expansion, and Applications,” q-alg/9711016.
- [28] B. Fedosov, “A Simple Geometrical Construction of Deformation Quantization,” J. Diff. Geom. **40** (1994) 213–238.
- [29] N. Boulanger, “A Weyl-covariant tensor calculus,” J. Math. Phys. **46** (2005) 053508, hep-th/0412314.
- [30] V. Didenko and E. Skvortsov, “Elements of Vasiliev theory,” 1401.2975.